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On 2-distance $(\Delta+4)$-coloring of planar graphs with girth five. (English) Zbl 08184176
Acta Math. Appl. Sin., Engl. Ser. 42, No. 2, 569-581 (2026).

A 2-distance coloring of a (simple) graph G is a proper vertex coloring such that any two vertices at distance at most two from each other receive distinct colors; equivalently, it is a proper coloring of the square G^2. The least number of colors needed for such a coloring is the 2-distance chromatic number of G, denoted $\chi_2(G)$. Trivially, $\chi_2(G) \geq \Delta(G)+1$. *G. Wegner* ["Graphs with given diameter and a coloring problem", Technical Report, University of Dortmund (1977)] conjectured that, for every planar graph G, $\chi_2(G) \leq 7$ when $\Delta(G)=3$, $\chi_2(G) \leq \Delta(G)+5$ when $4 \leq \Delta(G) \leq 7$, and $\chi_2(G) \leq \lfloor 3\Delta(G)/2 \rfloor + 1$ when $\Delta(G) \geq 8$. The case $\Delta(G)=3$ was settled by *C. Thomassen* [J. Comb. Theory, Ser. B 128, 192–218 (2018; Zbl 1375.05110)] and independently by *S. G. Hartke* et al. ["The chromatic number of the square of subcubic planar graphs", Preprint, `arXiv:1604.06504`], but otherwise the conjecture remains open. For a dynamic survey of the area, see [*D. W. Cranston*, Electron. J. Comb. DS25, 42 p. (2023; Zbl 1519.05072)].

In this paper, the author considers the case of planar graphs G of girth at least five. For this class, the bound $\chi_2(G) \leq \Delta(G)+4$ has previously been established under successively lower thresholds on $\Delta(G)$: for $\Delta(G) \geq 150$ by *W. Dong* and *W. Lin* [Discrete Appl. Math. 217, Part 3, 495–505 (2017; Zbl 1358.05101)], and for $\Delta(G) \geq 40$ by *Y. D. Jin* and *L. Y. Miao* [Acta Math. Appl. Sin., Engl. Ser. 38, No. 3, 540–548 (2022; Zbl 1492.05045)] as a consequence of an analogous statement for list 2-distance colorings. The main result of this paper is to lower the threshold on $\Delta(G)$ for which the bound $\Delta(G)+4$ is known: precisely, that $\chi_2(G) \leq \Delta(G)+4$ for any planar graph G with girth at least 5 and $\Delta(G) \geq 22$.

The proof is by the discharging method applied to a minimal counterexample. The author introduces a classification of vertices into "light" and "heavy". Writing $D(v) = \sum_{u \in N(v)} d(u)$, a vertex v is called light if $D(v) < \Delta(G)+4+n_2^3(v)$, where $n_2^3(v)$ is the number of 2-neighbors of v that are adjacent to a 3-vertex other than v; and heavy if $D(v) \geq \Delta(G)+4+n^l(v)$, where $n^l(v)$ is the number of light neighbors of v. A direct color-availability argument shows that every neighbor of a light vertex must be heavy, from which the author derives a number of forbidden configurations involving vertices of small degree. The hypothesis $\Delta(G) \geq 22$ enters through inequalities of the form $D(v) \geq \Delta(G)+4+n^l(v) \geq 26$ that are used to force certain neighbors of v to have sufficiently high degree.

The improvement is purely on the threshold; the bound $\Delta(G)+4$ itself is unchanged from the prior work of Jin-Miao [loc. cit.] and Dong-Lin [loc. cit.]. Closing the gap with the bound $\chi_2(G) \leq \Delta(G)+3$ for $\Delta(G) \geq 339$ due to *W. Dong* and *B. Xu* [J. Comb. Optim. 34, 1302–1322 (2017; Zbl 1373.05063)] remains an open problem.

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MSC:

05C15 Coloring of graphs and hypergraphs
05C10 Planar graphs; geometric and topological aspects of graph theory
05C12 Distance in graphs

Keywords:

coloring; 2-distance coloring; girth; distance; planar graph

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